A connectedness result for commuting diffeomorphisms of the interval

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Abstract

Let $\mathcal{D}^r_+[0,1]$, $r \geq 1$, denote the group of orientation-preserving $\mathcal{C}^r$ diffeomorphisms of $[0,1]$. We show that any two representations of $\mathbb{Z}^2$ in $\mathcal{D}^r_+[0,1]$, $r \geq 2$, are connected by a continuous path of representations of $\mathbb{Z}^2$ in $\mathcal{D}^1_+[0,1]$. We derive this result from the classical works by G. Szekeres and N. Kopell on the $\mathcal{C}^1$ centralizers of the diffeomorphisms of $[0,1)$ which are at least $\mathcal{C}^2$ and fix only 0.

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Let $\mathcal{D}^r_+[0,1]$, $r \geq 1$, denote the group of orientation-preserving $\mathcal{C}^r$ diffeomorphisms of the interval $[0,1]$. For any manifold M, the homomorphisms of $\pi_1 M$ to $\mathcal{D}^r_+[0,1]$, viewed as holonomy representations — and so, called below representations in $\mathcal{D}^r_+[0,1]$ —, describe $\mathcal{C}^r$ foliations of codimension 1 on $M \times [0,1]$ that are transverse to the $[0,1]$ factor and tangent to the boundary. In order to understand the topology of codimension 1 foliation spaces, we have therefore to study the topology of the space of representations of finitely presented/generated groups in $\mathcal{D}^r_+[0,1]$. The result we prove in this paper is a first step in this direction. Applied to the holonomy of torus leaves, it plays a crucial role in our work on deformations of codimension 1 foliations on 3-manifolds [Ey2]. The statement is:

Theorem A. *For $r \geq 2$, any two representations of $\mathbb{Z}^2$ in $\mathcal{D}^r_+[0,1]$ can be connected by a continuous path of representations of $\mathbb{Z}^2$ in $\mathcal{D}^1_+[0,1]$.*

Note that a representation of $\mathbb{Z}^2$ in $\mathcal{D}^r_+[0,1]$ is nothing but a pair of commuting $\mathcal{C}^r$ diffeomorphisms. Thus, the space of these representations will be regarded here as the subspace

$$\mathcal{R}^r = \left\{(f,g) \in (\mathcal{D}^r_+[0,1])^2 \mid f \circ g = g \circ f\right\} \subset (\mathcal{D}^r_+[0,1])^2$$

equipped with the induced topology, where $\mathcal{D}^r_+[0,1]$ is endowed with the usual $\mathcal{C}^r$ topology. The question we must answer can be phrased as follows:

Given two commuting diffeomorphisms $f, g \in \mathcal{D}^r_+[0,1]$, how to connect the pairs (f,g) and $(\mathrm{id},\mathrm{id})$ by a continuous path $t \in [0,1] \mapsto (f_t, g_t)$ in $\mathcal{R}^1$?

The key tools to handle this problem are classical results due to G. Szekeres and N. Kopell [Sz, Ko] (see Sections 1 and 2) about the $\mathcal{C}^1$ centralizers of the diffeomorphisms of $[0,1)$ which are at least $\mathcal{C}^2$ and fix only 0. More precisely, our construction proceeds as follows. The common fixed points of f and g form a closed set whose complement is a countable union of disjoint open intervals $(a,b) \subset [0,1]$. On each of these, Kopell's Lemma (Theorem 1) shows that f (resp. g) either coincides with the identity or has no fixed point (Lemma 6). Then the works by Szekeres and Kopell imply that the restrictions of f and g to $[a,b]$ belong either to a common $\mathcal{C}^1$ flow or to a common infinite cyclic group generated by some $\mathcal{C}^r$ diffeomorphism of $[a,b]$ (Lemma 6). In either case, it is easy to define the desired

pair (f_t, g_t) on $[a, b]$ (Lemma 10). Our main contribution in this article is to prove that all these pairs of local $\mathcal{C}^1$ diffeomorphisms fit together to yield a continuous path in $\mathcal{R}^1$ (Lemma 10). A useful tool at this point is a result of F. Takens [Ta] (Theorem 8) which shows that many adjacent subintervals (a, b) can be merged and treated as a single piece (Lemma 9), which makes it easier to check the global regularity.

Remarks. (i) Theorem A actually extends to representations of $\mathbb{Z}^k$ for any $k \in \mathbb{N}$ with exactly the same proof but heavier notations.

(ii) Unfortunately, Theorem A says nothing about the connectedness of $\mathcal{R}^1$ because $\mathcal{C}^1$ diffeomorphisms and their $\mathcal{C}^1$ centralizers are beyond the scope of the works of Szekeres and Kopell. On the other hand, it will become clear in the next section that our construction does not yield in general a path in $\mathcal{R}^r$. Thus, the question whether or not $\mathcal{R}^r$ is connected remains completely open for every integer r, $1 \leq r \leq \infty$.

Notations. For any $\mathcal{C}^k$ function g on an interval $I \subset \mathbb{R}$, we write

$$\|g\|_k = \sup \left\{ \left| D^m g(x) \right|,\ 0 \leq m \leq k,\ x \in I \right\} \in [0, +\infty].$$

Given a vector field ν on I, we make no difference between ν and the function $dx(\nu)$, where x is the coordinate in I. Thus, the pull-back $f^*\nu$ of ν by a diffeomorphism f of I is the function $\nu \circ f / Df$. Finally, we denote by f^k the k^{th} iterate of any diffeomorphism f of I, with $k \in \mathbb{Z}$.

1 The results of Szekeres and Kopell

Let $\mathcal{D}^r I$, $1 \leq r \leq \infty$, denote the group of $\mathcal{C}^r$ diffeomorphisms of an interval $I \subset \mathbb{R}$ and $\mathcal{D}^r_+ I$ the subgroup of those that preserve orientation. For $1 \leq k \leq r$, the $\mathcal{C}^k$ centralizer of an element $f \in \mathcal{D}^r I$ is

$$\mathcal{Z}^k_f = \{ g \in \mathcal{D}^k I \,;\, g \circ f = f \circ g \}.$$

Though this may be a quite complicated object in general, works by Szekeres and Kopell lead to a complete understanding of the case $k = 1$ and $r \geq 2$ when I is a semi-open interval and f has no interior fixed point. In this section, we recall their results and establish bounds that we use later in our argument. The original references are [Sz, Ko] but detailed proofs of Theorems 1 and 2 (and a lot more on the subject) can also be found in [Na] and [Yo].

Theorem 1 (Kopell's Lemma)**.** *Let f and g be two commuting diffeomorphisms of $[a, b) \subset \mathbb{R}$ which are of class $\mathcal{C}^2$ and $\mathcal{C}^1$, respectively. If f has no fixed point in (a, b) and g has at least one, then $g = \mathrm{id}$.*

Theorem 2 (Szekeres, Kopell)**.** *Let $f \in \mathcal{D}^r[a, b)$ be a diffeomorphism of $[a, b)$ fixing only a and assume $r \geq 2$. There exists a unique $\mathcal{C}^1$ vector field $\nu_f^{[a,b)}$ on*

$[a,b)$ *whose time-*1 *map exists and coincides with* f*. Moreover,* $\nu_f^{[a,b)}$ *is of class* $\mathcal{C}^{r-1}$ *on* (a,b) *and its flow, which is a one-parameter subgroup of* $\mathcal{D}^1[a,b)$*, equals the whole centralizer* $\mathcal{Z}_f^1$*.*

In this theorem, the existence part is due to Szekeres, and the uniqueness part follows from Kopell's Lemma. The vector field $\nu_f^{[a,b)}$ will be called the *Szekeres vector field of* f. It can be nicely expressed in the form:

$$\nu_f^{[a,b)} = \begin{cases} \lambda \lim\limits_{k\to+\infty} (f^k)^*\eta_0 & \text{if } f(x) < x \text{ for all } x \in (a,b), \\ \lambda \lim\limits_{k\to-\infty} (f^k)^*\eta_0 & \text{if } f(x) > x \text{ for all } x \in (a,b), \end{cases} \tag{1}$$

where $\eta_0 := (f - \mathrm{id})\partial_x$ is a $\mathcal{C}^{r-1}$ vector field on $[a,b)$, and $\lambda := \frac{\log Df(a)}{Df(a)-1}$, or 1 if $Df(a) = 1$. In other words, the proof of Szekeres' Theorem consists in showing that:

- the vector fields $\lambda(f^k)^*\eta_0$ converge in the $\mathcal{C}^1$ topology as k goes to $\pm\infty$ (depending on the sign of $f - \mathrm{id}$), and
- f is the time-1 map of the limit vector field.

Complete proofs of these assertions can be found in [Na] and [Yo]).

Remark 3. In general, one cannot expect the sequence $(f^k)^*\eta_0$ to converge in a stronger topology, even if the diffeomorphism (and thus every $(f^k)^*\eta_0$, $k \in \mathbb{Z}$) is $\mathcal{C}^\infty$. Indeed, F. Sergeraert constructed in [Se] a $\mathcal{C}^\infty$ diffeomorphism of $\mathbb{R}_+$ whose Szekeres vector field is not $\mathcal{C}^2$.

Expression (1) leads to the following estimates which control the Szekeres vector field in terms of the given diffeomorphism and reflect some continuous dependence (this continuity was studied more thoroughly by J.-C. Yoccoz in [Yo]; our bounds are established using arguments similar to his).

Lemma 4. *Let* $f \in \mathcal{D}^r[a,b)$ *be a diffeomorphism of* $[a,b)$ *fixing only* a*, with* $0 \le a < b \le 1$ *and* $r \ge 2$*. If* $\|f - \mathrm{id}\|_2 < \delta < 1$ *then the Szekeres vector field* $\nu := \nu_f^{[a,b)}$ *satisfies*

$$\sup_{(a,b)} \left| \log \frac{\nu}{f - \mathrm{id}} \right| < u(\delta) \quad \textit{and} \quad \sup_{[a,b)} |D\nu| < u(\delta)$$

for some universal continuous function $u\colon [0,1) \to \mathbb{R}$ *(independent of* f*,* a *and* b*) vanishing at* 0*. As a consequence, there exists another universal continuous function* v *vanishing at* 0 *such that*

$$\left\| f^t - \mathrm{id} \right\|_1 < t\, v(\delta) \quad \textit{for all } t \in [0,1],$$

where $\{f^t\}_{t\in\mathbb{R}}$ *denotes the flow of* ν*, with* $f^1 = f$*.*

Proof. We consider the case of a contracting diffeomorphism f. Let η_k denote the $\mathcal{C}^{r-1}$ vector field $(f^k)^*\eta_0$ on $[a,b)$ for all $k \in \mathbb{N}$, where $\eta_0 = (f - \mathrm{id})\partial_x$, and define $\theta\colon (a,b) \to \mathbb{R}$ by $\theta := \log \frac{\eta_1}{\eta_0}$. For all $x \in (a,b)$,

$$\theta(x) = \log \frac{f^2(x) - f(x)}{Df(x)\,(f(x) - x)} = \log \left(\int_0^1 Df\big(x + s(f(x) - x)\big)\, ds \right) - \log(Df(x)).$$

Thus, since Df is $\mathcal{C}^{r-1}$ and positive on $[a,b)$, the map θ extends to a $\mathcal{C}^{r-1}$ map on $[a,b)$. One can also write

$$\theta(x) = \log Df(x_0) - \log Df(x) \quad \text{for some} \quad x_0 \in [f(x), x].$$

Since $\log Df$ is $\mathcal{C}^{r-1}$ with $r \geq 2$, this implies

$$|\theta(x)| \leq \|D \log Df\|_0 \, |x_0 - x| = \left\| \frac{D^2 f}{Df} \right\|_0 |x_0 - x| \leq \frac{\delta}{1-\delta}(x - f(x)),$$

according to our hypothesis on $\|f - \mathrm{id}\|_2$. Now

$$\log \frac{\eta_{k+1}}{\eta_k} = \log \frac{(f^k)^*\eta_1}{(f^k)^*\eta_0} = \log \frac{\eta_1 \circ f^k}{\eta_0 \circ f^k} = \theta \circ f^k,$$

so

$$\left| \log \frac{\eta_j}{\eta_i}(x) \right| = \left| \sum_{k=i}^{j-1} \theta \circ f^k \right| \leq \frac{\delta}{1-\delta} \sum_{k=i}^{j-1} (f^k(x) - f^{k+1}(x)) \leq \frac{\delta}{1-\delta}. \tag{2}$$

Taking $i = 0$ and $j \to \infty$, this gives the first bound of the lemma since

$$\lambda = \frac{\log Df(a)}{Df(a) - 1} < \frac{|\log(1-\delta)|}{\delta} \xrightarrow[\delta \to 0]{} 1.$$

The second estimate relies on the following calculation, where $Lg := D^2 g / Dg$ for any $\mathcal{C}^2$ diffeomorphism g:

$$\begin{aligned}
D\eta_k = D\left(\frac{\eta_0 \circ f^k}{Df^k} \right) &= D\eta_0 \circ f^k + (\eta_0 \circ f^k) D\left(\frac{1}{Df^k} \right) \\
&= D\eta_0 \circ f^k - (\eta_k Df^k) \frac{D^2 f^k}{(Df^k)^2} \\
&= D\eta_0 \circ f^k - Lf^k \eta_k \\
&= D\eta_0 \circ f^k - \sum_{i=0}^{k-1} (Lf \circ f^i)\, Df^i \; \eta_k \\
&= D\eta_0 \circ f^k - \sum_{i=0}^{k-1} (Lf \circ f^i)(f^{i+1} - f^i) \frac{\eta_k}{\eta_i}.
\end{aligned}$$

The desired bound follows easilly, using (2) and the fact that $D\eta_0$ equals $Df - 1$. $\square$

2 Rational and irrational connected components

Definition 5. Let $f, g \in \mathcal{D}^r_+[0,1]$ be two commuting diffeomorphisms and denote by $F \subset [0,1]$ the set of their common fixed points. Let us say that a connected component (a,b) of the open set $U = [0,1] \setminus F$ is *rational* or *irrational* depending on whether or not there exist relatively prime integers $p, q \in \mathbb{Z}$ such that f^p and g^q coincide on (a,b).

For example, a component (a,b) on which f or g induces the identity is rational, for 0 and 1 are relatively prime.

Lemma 6. *Let $f, g \in \mathcal{D}^r_+[0,1]$ be two commuting diffeomorphisms, F the set of their common fixed points and (a,b) a connected component of $U = [0,1] \setminus F$.*

0. *If $f\,|_{[a,b]}$ differs from the identity, then f has no fixed point in (a,b) and thus defines two Szekeres vector fields: $\nu_f^{[a,b)}$ on $[a,b)$ and $\nu_f^{(a,b]}$ on $(a,b]$.*
1. *If the component (a,b) is rational, there exist a diffeomorphism $h \in \mathcal{D}^r_+[a,b]$ and some relatively prime integers $p, q \in \mathbb{Z}$ such that $f\,|_{[a,b]} = h^q$ and $g\,|_{[a,b]} = h^p$. Moreover, if $f|_{[a,b]}$ is not the identity, $h|_{(a,b)}$ coincides with the time-$1/q$ maps of both Szekeres vector fields $\nu_f^{[a,b)}$ and $\nu_f^{(a,b]}$.*
2. *If the component (a,b) is irrational, $\nu_f^{[a,b)}$ and $\nu_f^{(a,b]}$ coincide on (a,b). Thus, there is a $\mathcal{C}^1$ vector field $\nu_f^{[a,b]}$ on $[a,b]$ whose time-1 map is $f\,|_{[a,b]}$, and $g\,|_{[a,b]}$ is the time-τ map of this vector field for some $\tau \in \mathbb{R} \setminus \mathbb{Q}$.*

Remark 7. According to Kopell [Ko], for a generic $\mathcal{C}^r$ diffeomorphism f of $[a,b]$ with no fixed points in (a,b), the Szekeres vector fields $\nu_f^{[a,b)}$ and $\nu_f^{(a,b]}$ do not coincide on (a,b). In other words, $\nu_f^{[a,b)}$ does not extend to a $\mathcal{C}^1$ vector field on $[a,b]$. So one really needs to handle the rational case seperately.

As for irrational components, one might think that having $\mathcal{C}^r$ time-t maps for a dense subset $\mathbb{Z} + \tau\mathbb{Z} \subset \mathbb{R}$ of times t would force a vector field to be $\mathcal{C}^{r-1}$. But this is not true, according to [Ey1]. Thus, the diffeomorphisms obtained with our method (cf. Lemma 10) are only $\mathcal{C}^1$ in general.

Proof. Suppose f has a fixed point c in (a,b). The sequence $(g^n(c))_{n\in\mathbb{Z}}$ stays in (a,b), consists of fixed points of f (for f and g commute) and is monotone (for c cannot be a fixed point of both f and g by definition of (a,b)). Thus, this sequence converges at both ends towards points which necessarily lie in F, and hence are a and b. Therefore g has no fixed point in (a,b) and Kopell's Lemma (Theorem 1) shows that f is the identity on $[a,b]$, which concludes the first point.

If (a,b) is a rational component, there exist relatively prime numbers $p, q \in \mathbb{Z}$ such that f^p and g^q coincide on $[a,b]$. Then, writing $h = (f^s g^r)\,|_{[a,b]}$ where $pr + qs = 1$, $r, s \in \mathbb{Z}$, one gets the desired relations $f\,|_{[a,b]} = h^q$ and $g\,|_{[a,b]} = h^p$. If $f\,|_{[a,b]}$ is not the identity, Theorem 2 assures that the restrictions of f, g and

h to $[a,b)$ (resp. $(a,b]$) belong to the flow of the Szekeres vector field $\nu_f^{[a,b)}$ (resp. $\nu_f^{(a,b]}$). The corresponding time for h is $1/q$ since $h^q = f \mid_{[a,b]}$.

Now suppose the component (a,b) is irrational. Denote by $\{f_a^t\}_{t\in\mathbb{R}}$ and $\{f_b^t\}_{t\in\mathbb{R}}$ the flows of $\nu_f^{[a,b)}$ and $\nu_f^{(a,b]}$, respectively, and fix a point $c \in (a,b)$. The diffeomorphisms $\psi_a, \psi_b \colon \mathbb{R} \to (a,b)$ defined by $\psi_a(t) = f_a^t(c)$ and $\psi_b(t) = f_b^t(c)$, respectively, conjugate $f_a^\tau \mid_{(a,b)}$ and $f_b^\tau \mid_{(a,b)}$, $\tau \in \mathbb{R}$, to the translation $T_\tau \colon t \mapsto t+\tau$:

$$T_\tau = \psi_a^{-1} \circ f_a^\tau \circ \psi_a = \psi_b^{-1} \circ f_b^\tau \circ \psi_b.$$

In particular,

$$T_1 = \psi_a^{-1} \circ f \circ \psi_a = \psi_b^{-1} \circ f \circ \psi_b,$$

so

$$T_1 = (\psi_a^{-1} \circ \psi_b) \circ T_1 \circ (\psi_b^{-1} \circ \psi_a).$$

In other words, $\psi_b^{-1} \circ \psi_a$ is a diffeomorphism of $\mathbb{R}$ which commutes with the unit translation T_1.

According to Theorem 2, $g \mid_{[a,b)}$ (resp. $g \mid_{(a,b]}$) coincides with $f_a^{\tau_a}$ (resp. $f_b^{\tau_b}$) for some time τ_a (resp. τ_b). But then

$$T_{\tau_a} = \psi_a^{-1} \circ g \circ \psi_a \quad \text{and} \quad T_{\tau_b} = \psi_b^{-1} \circ g \circ \psi_b$$

so

$$T_{\tau_a} = (\psi_a^{-1} \circ \psi_b) \circ T_{\tau_b} \circ (\psi_b^{-1} \circ \psi_a).$$

All four diffeomorphisms of $\mathbb{R}$ in this last equality commute with the unit translation. Therefore, invariance of the rotation number under conjugacy implies that $\tau_a = \tau_b =: \tau$. This number has to be irrational, for if $\tau = p/q$ then f^p clearly coincides with g^q on (a,b). But if so the diffeomorphism $\psi_b^{-1} \circ \psi_a$ commutes with both the unit translation and an irrational translation, and hence it must itself be a translation. Since it fixes the origin (by construction), it is in fact the identity. This means that the flows of $\nu_f^{[a,b)}$ and $\nu_f^{(a,b]}$ coincide on (a,b), so these vector fields are equal on (a,b). $\square$

We will now see (Lemma 9) that the type of the components of $[0,1] \setminus F$ is in fact constant on the components of $[0,1] \setminus F_0$, where $F_0 \subset F$ is the set where both f and g are $\mathcal{C}^r$-tangent to the identity. This is a straightforward consequence of a theorem of Takens [Ta] (extended by Yoccoz in the finite differentability case [Yo]) which can be stated as follows:

Theorem 8 (Takens)**.** *Let $f \in \mathcal{D}_+^r(a,b)$, $r \geq 2$, be a diffeomorphism with a unique fixed point c. If f is not $\mathcal{C}^r$-tangent to the identity at c, the Szekeres vector fields $\nu_f^{(a,c]}$ and $\nu_f^{[c,b)}$ are $\mathcal{C}^{r-1}$ and fit together to yield a $\mathcal{C}^{r-1}$ vector field on (a,b) (whose time-1 map is f). Furthermore, any $\mathcal{C}^r$ diffeomorphism of (a,b) commuting with f and fixing c coincides with the time-τ map of this vector field for some $\tau \in \mathbb{R}$.*

Lemma 9. *Let $f, g \in \mathcal{D}^r_+[0,1]$ be two commuting diffeomorphisms, F the set of their common fixed points, $F_0 \subset F$ the subset of those where both f and g are $\mathcal{C}^r$-tangent to the identity, and (a,b) a connected component of $U_0 = [0,1] \setminus F_0$.*

1. *If (a,b) contains a rational component of $U = [0,1] \setminus F$, there exists a diffeomorphism $h \in \mathcal{D}^r_+[a,b]$ and relatively prime integers $p, q \in \mathbb{Z}$ such that $f\,|_{[a,b]} = h^q$ and $g\,|_{[a,b]} = h^p$.*

2. *If (a,b) contains an irrational component of U, there exists a vector field $\nu_f^{[a,b]}$ of class $\mathcal{C}^1$ on $[a,b]$, $\mathcal{C}^{r-1}$ on (a,b), $\mathcal{C}^1$-flat at the boundaries and whose time-1 and τ maps, for some $\tau \in \mathbb{R} \setminus \mathbb{Q}$, are $f\,|_{[a,b]}$ and $g\,|_{[a,b]}$, respectively.*

From now on, a *rational* (resp. *irrational*) component will be any component of $U_0 = [0,1] \setminus F_0$ which contains (only) rational (resp. irrational) components of $[0,1] \setminus F$.

3 Regularity at the degenerated fixed points

Theorem A is a straightforward consequence of the following lemma.

Lemma 10. *Let $f, g \in \mathcal{D}^r_+[0,1]$ be two commuting diffeomorphisms, F_0 the set of the common fixed points where both are $\mathcal{C}^r$-tangent to the identity, U_0 the complement $[0,1] \setminus F_0$ and ν the vector field on $[0,1]$ equal to $\nu_f^{[a,b]}$ on the closure of any irrational component (a,b) of U_0 and to 0 elsewhere. For any $t \in [0,1]$, define $f_t, g_t \colon [0,1] \to [0,1]$ as follows:*

- *on F_0, set $f_t = f = \mathrm{id}$ and $g_t = g = \mathrm{id}$;*
- *on each rational component of U_0 where $f = h^q$ and $g = h^p$ for relatively prime integers $p, q \in \mathbb{Z}$, set $f_t = h_t^q$ and $g_t = h_t^p$, where $h_t = (1-t)h + t\mathrm{id}$;*
- *on each irrational component of U_0 where f and g coincide with the flow of ν at times 1 and $\tau \in \mathbb{R} \setminus \mathbb{Q}$, respectively, let f_t and g_t be the flow maps of ν at times $(1-t)$ and $(1-t)\tau$, respectively.*

Then the arc $t \in [0,1] \mapsto (f_t, g_t)$ is a continuous path in the space $\mathcal{R}^1 \subset (\mathcal{D}^1_+[0,1])^2$ of commuting $\mathcal{C}^1$ diffeomorphisms.

Proof. The maps f_t and g_t clearly commute for all $t \in [0,1]$. Since f and g play symmetric roles in the construction, it is sufficient to prove that $t \mapsto f_t$ is a continuous path in $\mathcal{D}^1_+[0,1]$. First observe that, for all $t \in [0,1]$, the map f_t is a homeomorphism of $[0,1]$ and, according to Lemma 9, induces a $\mathcal{C}^{r-1}$ diffeomorphism of U_0. Now let us prove that f_t is differentiable at any point c of F_0, with derivative equal to 1.

Let $c \in F_0$. By construction,

$$|f_t(x) - x| \leq |f(x) - x| \quad \text{for all } (t,x) \in [0,1] \times [0,1].$$

If $x \neq c$,

$$\begin{aligned}\left|\frac{f_t(x)-f_t(c)}{x-c}-1\right| &= \left|\frac{(f_t(x)-x)-(f_t(c)-c)}{x-c}\right| \\ &= \left|\frac{f_t(x)-x}{x-c}\right| \\ &\leq \left|\frac{f(x)-x}{x-c}\right| \\ &= \left|\frac{(f(x)-x)-(f(c)-c)}{x-c}\right|.\end{aligned}$$

Since $c \in F_0$, the map $f - \mathrm{id}$ is $\mathcal{C}^1$-flat at c, so the above quantity tends to 0 as x goes to c. Therefore, f_t admits a derivative at c which is equal to 1.

To conclude the proof, one needs to check that the map $\Psi\colon (t,x) \mapsto Df_t(x)$ (now well-defined) is continuous on $[0,1]\times[0,1]$. For every component (c,d) of U_0, continuity on $[0,1]\times[c,d]$ follows readily from Lemma 9. In particular, the limit of $Df_s(x)$ as (s,x) approaches (t,c) in $[0,1]\times[c,1]$ exists and equals $Df_t(c) = 1$. Now assume that c is an accumulation point of F_0 on the right side. For all $\delta > 0$, there exists a point $d \in F_0 \cap (c,1]$ such that

$$\left\| f - \mathrm{id} \mid_{[c,d]} \right\|_2 < \delta.$$

Let $(s,x) \in [0,1]\times[c,d]$. If x belongs to F_0 then $Df_s(x) = 1$. If x belongs to an irrational component (a,b) of $U_0 \cap [c,d]$ then $f_s \mid_{[a,b]}$ belongs to the flow of ν, so $\nu \circ f_s(x) = Df_s(x)\,\nu(x)$. If x is not a fixed point of f (*i.e.* a zero of ν) then

$$\begin{aligned}|Df_s(x)-1| &= \left|\frac{\nu\circ f_s(x)-\nu(x)}{\nu(x)}\right| \\ &\leq \sup_{(a,b)} |D\nu| \left|\frac{f_s(x)-x}{\nu(x)}\right| \\ &\leq \sup_{(a,b)} \left|D\nu_f^{[a,b]}\right| \left|\frac{f(x)-x}{\nu(x)}\right| \leq u(\delta)e^{u(\delta)} \quad \text{according to Lemma 4.}\end{aligned}$$

This upper bound still holds for all $x \in [a,b]$ since one already knows that Ψ is continuous on $[0,1]\times[a,b]$.

Assume now that x belongs to a rational component (a,b) of $U_0 \cap [c,d]$ where $f = h^q$ and $g = h^p$, with $p, q \in \mathbb{Z}$ relatively prime. If q is zero, $f_s = \mathrm{id}$ on $[a,b]$ so $Df_s(x) = 1$. If q is nonzero, Lemma 4 bounds $\|h - \mathrm{id}\|_1$ by $\frac{1}{q}v(\delta)$. In particular,

for δ small enough, $\|h - \mathrm{id}\|_1 < 1/2$. Since $f_s = h_s^q$ on $[a,b]$,

$$\begin{aligned}
|\log Df_s(x)| = |\log D(h_s^q)(x)| &= \left|\sum_{i=0}^{q-1} \log Dh_s\big(h_s^i(x)\big)\right| \\
&\leq \sum_{i=0}^{q-1} \left|\log\Big(1 + (1-s)\Big(Dh\big(h_s^i(x)\big) - 1\Big)\Big)\right| \\
&\leq \sum_{i=0}^{q-1} 2(1-s)\,\|h - \mathrm{id}\|_1 \leq 2v(\delta).
\end{aligned}$$

Thus, $Df_s(x)$ tends to $1 = Df_s(c)$ as (s,x) goes to (t,c) in $[0,1] \times [c,1]$. Similarly, $Df_s(x)$ tends to 1 as (s,x) goes to (t,c) in $[0,1] \times [0,c]$. This proves the continuity of Ψ at every point in $[0,1] \times F_0$, and thus on the whole of $[0,1] \times [0,1]$. □

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